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On the Propagation of an Arbitrary Electro-Magnetic Disturbance, on Spherical Waves of Light and the Dynamical Theory of Diffraction.

BY PROFESSOR H. A. ROWLAND.

INTRODUCTION.

In the year 1849 the great paper of Stokes "On the Dynamical Theory of Diffraction" was read before the Cambridge Philosophical Society, and this has remained until the present day the standard upon this important subject.

The method of Stokes was based upon the old elastic solid theory of light, and gave the following conclusions:

First. That when the incident light was plane polarized, the diffracted light from a small orifice was also plane polarized in such a manner that the displacement was in the same plane as that of the medium at the orifice. So that if a sphere was drawn with the orifice as a center and meridians drawn on the sphere with the axis in the direction of the vibration at the orifice, then these meridians represented the direction of displacement in the diffracted light.

Second. The intensity of the polarized light was represented as follows: Let δ represent the angle between the incident ray prolonged and the diffracted ray, and let ϕ be the angle between the diffracted ray and the direction of the displacement at the orifice. Then the intensity of the diffracted light around a very small orifice will be proportional to

$$(1 + \cos \delta)^2 \sin^2 \phi .$$

The presence of the term in ϕ indicates that the intensity of the diffracted light at a given point varies as one rotates the plane of polarization. In ordinary light the intensity varies as $1 + \cos^2 \delta$.

Stokes and others have attempted to determine the relation between the plane of polarization and the direction of displacement by means of the first relation, but they have not agreed with one another, and this want of agreement has usually been assigned to the fact that the gratings used have been ruled on glass rather than in free space, as the equations indicate.

On examining this question from the point of view of the electro-magnetic theory of light, I have been led to entirely different results. But as the elastic solid theory for an incompressible solid must agree with the electro-magnetic theory, I have been led to examine the theory of Stokes, and believe that I have now discovered an error, which, if it were corrected, would lead to my result.

The results which I reach are as follows:

First. The plane of polarization of the diffracted light is determined as follows: Draw a sphere around the orifice and mark the point on the sphere where the incident light *enters* it. Through this point draw circles on the sphere whose planes are parallel to the electrostatic displacement at the orifice, and these circles give the direction of the electrostatic disturbance in the diffracted beam. Repeat the same for the magnetic disturbance, which is at right angles to the electrical disturbance, and the circles indicate the direction of the magnetic disturbance in the diffracted beam. These two systems of circles are orthogonal to one another, as they should be.

Second. The *intensity* of the diffracted light around a very small orifice *is symmetrical around the incident ray prolonged* and is proportional to

$$(1 + \cos \delta)^2,$$

and the same expression applies to ordinary light.

Although the theory of diffraction forms the most interesting part of my paper, yet I have thought it worth while to treat of the general problem of spherical waves of light, which I have not seen considered anywhere else. The method is similar to that used in sound and the theory of heat conduction.

SPHERICAL WAVES.

Let F_0, G_0, H_0; F_1, G_1, etc., be a system of vectors derived from one another by the equations

$$F_{m+1} = \frac{dH_m}{dy} - \frac{dG_m}{dz}$$

$$G_{m+1} = \frac{dF_m}{dz} - \frac{dH_m}{dx}$$

$$H_{m+1} = \frac{dG_m}{dx} - \frac{dF_m}{dy}.$$

Then, as is well known, all this system of vectors necessarily satisfy the equation of continuity

$$\frac{dF_m}{dx}+\frac{dG_m}{dy}+\frac{dH_m}{dz}=0,$$

except the primitive ones F_0, G_0, H_0, and even these can be made to satisfy the equation by the addition of terms of the form $\frac{dJ}{dx}$, $\frac{dJ}{dy}$, $\frac{dJ}{dz}$.

The equations of light-waves either on the electro-magnetic theory or the elastic solid theory are of the form

$$\begin{aligned}
\frac{d^2F}{dt^2}&=V^2\left\{\Delta^2F-\frac{dJ}{dx}\right\}+v^2\frac{dJ}{dx}\\
\frac{d^2G}{dt^2}&=V^2\left\{\Delta^2G-\frac{dJ}{dx}\right\}+v^2\frac{dJ}{dy}\\
\frac{d^2H}{dt^2}&=V^2\left\{\Delta^2H-\frac{dJ}{dz}\right\}+v^2\frac{dJ}{dz}\\
J&=\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\\
\Delta^2&=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.
\end{aligned}$$

In the electro-magnetic theory v is always zero. In the elastic solid theory the terms in v give the wave of normal disturbance like that of sound, and can nearly always be omitted, as the transverse and normal waves are immediately separated when v is different from V, as it always must be. As we wish to use the derived functions only, the terms in J can also be omitted, as Maxwell has shown, and hence we can write on either theory

$$\begin{aligned}
\frac{d^2F}{dt^2}&=V^2\Delta^2F\\
\frac{d^2G}{dt^2}&=V^2\Delta^2G\\
\frac{d^2H}{dt^2}&=V^2\Delta^2H
\end{aligned}$$

V indicates the velocity of a plane wave in the medium.

Each one of these equations is the same as that of sound and can be solved in a similar manner. If we have solutions of these equations F_0, G_0, H_0, which do not satisfy the equation of continuity, then we can get other solutions F_m, G_m, H_m from these which *shall* satisfy this condition. Denote $\sqrt{-1}$ by i and let $V_{-(n+1)}$ be a solid harmonic. Then as $V_{-(n+1)}$ is a homogeneous function of x, y and z of the degree $-(n+1)$, we have

$$x\frac{dV_{-(n+1)}}{dx}+y\frac{dV_{-(n+1)}}{dy}+z\frac{dV_{-(n+1)}}{dz}=-(n+1)V_{-(n+1)}$$

and

$$\Delta^2 V_{-(n+1)}=0.$$

Make

$$F_0=C_n\rho^n V_{-(n+1)}\varepsilon^{(a-ib)(\rho-Vt)},$$

where C is a function of ρ of a complex form

$$C_n=A_n-iB_n$$

$$\rho^2=x^2+y^2+z^2.$$

This form has the advantage that by the addition of another term of the same form in which i has a negative value, the sum will reduce to a real form with circular arcs instead of powers of ε; a is introduced for generality. Substituting this form we have for the determination of C,* the equation

$$\frac{d^2C_n}{d\rho^2}+2(a-ib)\frac{dC_n}{d\rho}-\frac{n(n+1)}{\rho^2}C_n=0.$$

Writing $c=a-ib$ we readily find, as others have found before for this equation,

$$C_n=C_0\left\{1-\frac{n}{2}\,\frac{n+1}{c\rho}+\frac{n(n^2-1^2)}{2.4}\,\frac{n+2}{c^2\rho^2}-\frac{n(n^2-1^2)(n^2-2^2)}{2.4.6}+\text{etc.}\right\}$$

The series ends after $n+1$ terms, and so we can put it in the form

$$C_n=(-1)^n(1.3.5\ldots(2n-1))\frac{C_0}{c^n\rho^n}\left\{1-c\rho+\frac{2}{1.2}\,\frac{n-1}{2n-1}c^2\rho^2\right.$$
$$\left.-\frac{2.2}{1.2.3}\,\frac{(n-1)(n-2)}{(2n-1)(2n-2)}c^3\rho^3+\text{etc.}\right\}$$

The following properties of the functions C_n are useful. Writing $c=a-ib$, we have

$$C_n=\frac{c\rho}{2n+1}\{C_{n-1}-C_{n+1}\}$$

Making

$$C_n=D_n\rho^{\frac{1}{2}}\varepsilon^{-\rho(a-ib)},$$

we obtain

$$\frac{d^2D_n}{d\rho^2}+\frac{1}{\rho}\,\frac{dD_n}{d\rho}-D_n\left((a-ib)^2+\frac{(n+\frac{1}{2})^2}{\rho^2}\right)=0.$$

If we replace ρ by a new variable equal to $i(a-ib)\rho$, this reduces to the equation of Bessel's functions: hence we can write

$$D_n=J_{(n+\frac{1}{2})}(b+ia)\rho,$$

$$F_0=\{J_{(n+\frac{1}{2})}(b+ia)\rho\}\rho^{n+\frac{1}{2}}V_{-(n+1)}\varepsilon^{-Vt(a-ib)},$$

with similar terms for G_0 and H_0.

Hence

$$C_n=C_0'\rho^{\frac{1}{2}}\varepsilon^{-\rho(a-ib)}J_{(n+\frac{1}{2})}(b+ia)\rho,$$

where C_0' is a function of n to be determined, and one must add a corresponding term with $-i$ in place of $+i$.

$$\frac{dC_n}{d\rho} = c\left\{-C_n + \frac{1}{2n+1}[(n+1)C_{n-1} + nC_{n+1}]\right\} = cC_{n-1} - C_n\left(\frac{n}{\rho} + c\right),$$

$$C_{n+1} = C_n\left(1 - \frac{n+1}{c\rho}\right) + \frac{1}{c}\frac{dC_n}{d\rho}.$$

Putting $c\rho = cp\sqrt{1+2s}$ we have

$$\frac{d^{n+1}}{ds^{n+1}}\varepsilon^{c\rho} = \frac{c^{n+1}p^{2n+2}}{\rho^{n+1}}C_n\varepsilon^{c\rho}.$$

Extending the notation so that we should write $C_n(\rho)$ in the place of C_n as above, we have, if we put $pcs = q$

$$\varepsilon^{c\rho} = \varepsilon^{cp}\left\{1 + \frac{q}{1}C_0 + \frac{q^2}{1.2}C_1(p) + \frac{q^3}{1.2.3}C_2(p) + \text{etc.}\right\}$$

$$\frac{C_n(\rho)}{\rho^{n+1}}\varepsilon^{c\rho} = \frac{\varepsilon^{cp}}{p^{n+1}}\left\{C_n(p) + \frac{q}{1}C_{n+1}(p) + \frac{q^2}{1.2}C_{n+2}(p) + \text{etc.}\right\}$$

These expansions are useful in many calculations from the equations, and I believe they are given here for the first time, together with the differential expression from which they are derived.

Hence we have obtained a complete solution of one of the equations. Now the general value of $V_{-(n+1)}$ is

$$V_{-(n+1)} = (-1)^n \frac{d^n}{dh_1, dh_2, \ldots dh_n}\frac{1}{\rho}.$$

By varying the axes h_1, h_2, h_3, etc., the value and form of $V_{-(n+1)}$ will change, and can thus have many values while C remains unaltered. Select three of these forms and we can thus write

$$F_0 = C_n\rho^n V'_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y'_n\varepsilon^{c(\rho - Vt)},$$

$$G_0 = C_n\rho^n V''_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y''_n\varepsilon^{c(\rho - Vt)},$$

$$H_0 = C_n\rho^n V'''_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y'''_n\varepsilon^{c(\rho - Vt)},$$

where Y'_n, Y''_n, Y'''_n are surface harmonics.

These do not satisfy the equation of continuity, but the series of derived vectors F_m, G_m, H_m satisfy both the equation of continuity and the equations of light.

One of the best forms for these spherical harmonics is to assume one of the axes in each of them parallel to one of the coordinate axes. We can then write

$$F_0 = C_n\rho^n\frac{dV_{-n}}{dx}\varepsilon^{c(\rho - Vt)},$$

$$G_0 = C_n\rho^n\frac{dV_{-n}}{dy}\varepsilon^{c(\rho - Vt)},$$

$$H_0 = C_n\rho^n\frac{dV_{-n}}{dz}\varepsilon^{c(\rho - Vt)}.$$

In this case the vector represented by the components F_0, G_0, H_0 is perpendicular to the surface $V_{-n} =$ constant.

Now the components of this vector do not satisfy the equation of continuity, and so, although it may represent the vector potential on the electro-magnetic theory, it cannot represent either the electric current or magnetic force, and is almost without direct use on the elastic solid theory. Hence the derived vectors are of more general use. These are

$$F_1 = cC_{n-1}\rho^{n-1}\left\{y\frac{dV_{-n}}{dz} - z\frac{dV_{-n}}{dy}\right\}\varepsilon^{c(\rho-Vt)},$$

$$G_1 = cC_{n-1}\rho^{n-1}\left\{z\frac{dV_{-n}}{dx} - x\frac{dV_{-n}}{dz}\right\}\varepsilon^{c(\rho-Vt)},$$

$$H_1 = cC_{n-1}\rho^{n-1}\left\{x\frac{dV_{-n}}{dy} - y\frac{dV_{-n}}{dx}\right\}\varepsilon^{c(\rho-Vt)}.$$

$$F_2 = -c^2F_0 + \frac{dJ_0}{dx},$$

$$G_2 = -c^2G_0 + \frac{dJ_0}{dy},$$

$$H_2 = -c^2H_0 + \frac{dJ_0}{dz}.$$

Where

$$J_0 = \frac{dF_0}{dx} + \frac{dG_0}{dy} + \frac{dH_0}{dz} = -\frac{n}{\rho}V_{-n}\frac{d}{d\rho}\{C_n\rho^n\varepsilon^{c(\rho-Vt)}\},$$

$$J_0 = -ncC_{n-1}\rho^{n-1}V_{-n}\varepsilon^{c(\rho-Vt)},$$

$$F_3 = -c^2F_1, \text{ etc.}$$

The remaining derived vectors are simply repetitions of these with only change in the constants.

These derived vectors can represent the vector potential, the electric displacement, the electric current, the magnetic displacement, or the magnetic force in the electro-magnetic theory, and the displacement, the velocity of the particles or the rotational displacement or velocity of rotation on the elastic solid theory.

The most interesting case is that of symmetry around an axis, say X. In this case

$$V_{-n} = \frac{d^n}{dx^n}\frac{1}{\rho} = \frac{1}{\rho^n}\frac{(-1)^n}{1.2.3\ldots n}Q_{n-1},$$

where Q_n is a zonal surface harmonic.

If α is the angle measured around the axis of x from y and θ that made with the same axis, and we put $\rho\sin\theta = r$, we can then write, when m is even

$$R_m = \sqrt{G_m^2 + H_m^2},$$
$$N_{m+1} = \sqrt{G_{m+1}^2 + H_{m+1}^2},$$
$$M_{m+1} = N_{m+1} r$$

$$F_m = -\Delta^2 F_{m-2} = +\frac{1}{r}\frac{dM_{m-1}}{dr},$$
$$R_m = -\Delta^2 R_{m-2} - \frac{R_{m-2}}{r^2} = -\frac{1}{r}\frac{dM_{m-1}}{dx},$$

$$F_{m+1} = 0,$$
$$N_{m+1} = -\frac{dF_m}{dr} + \frac{dR_m}{dx} = -\Delta^2 N_{m-1} + \frac{N_{m-1}}{r^2} = -\frac{1}{r}\left\{\frac{d^2 M_{m-1}}{dx^2} + \frac{d^2 M_{m-1}}{dr^2} - \frac{1}{r}\frac{dM_{m-1}}{dr}\right\}.$$

The equation of continuity becomes

$$\frac{d(F_m r)}{dx} + \frac{d(R_m r)}{dr} = 0.$$

These can be expressed in terms of ρ and θ, as follows:

Let Θ_m, P_m and N_m be the components in the direction θ, ρ and α respectively. Then we have

$$\frac{d}{dr} = \sin\theta\frac{d}{d\rho} - \frac{\cos\theta\sin\theta}{\rho}\frac{d}{d\mu},$$
$$\frac{d}{dx} = \cos\theta\frac{d}{d\rho} + \frac{\sin^2\theta}{\rho}\frac{d}{d\mu},$$

$$\Theta_m = -F_m\sin\theta + R_m\cos\theta,$$
$$P_m = F_m\cos\theta + R_m\sin\theta,$$

$$\Theta_m = -\frac{1}{\rho\sin\theta}\frac{dM_{m-1}}{d\rho},$$
$$P_m = -\frac{1}{\rho^2}\frac{dM_{m-1}}{d\mu},$$
$$N_m = 0,$$

$$\Theta_{m+1} = 0,$$
$$P_{m+1} = 0,$$
$$N_{m+1} = -\frac{1}{\rho\sin\theta}\left\{\frac{d^2 M_{m-1}}{d\rho^2} + \frac{\sin^2\theta}{\rho^2}\frac{d^2 M_{m-1}}{d\mu^2}\right\}.$$

The equation of motion of light becomes in this case

$$\frac{1}{V^2}\frac{d^2 M_{m-1}}{dt^2} = \frac{d^2 M_{m-1}}{d\rho^2} + \frac{\sin^2\theta}{\rho^2}\frac{d^2 M_{m-1}}{d\mu^2},$$

whence
$$M_{m+1} = N_{m+1}\,\rho\sin\theta = -\frac{1}{V^2}\frac{d^2 M_{m-1}}{dt^2}.$$

Now, in this case of symmetry we readily find
$$F_0 = C_n\rho^n\frac{dV_{-n}}{dx}\varepsilon^{c(\rho-Vt)},$$
$$R_0 = C_n\rho^n\frac{dV_{-n}}{dr}\varepsilon^{c(\rho-Vt)},$$
and for convenience put $U = C_n\rho^n\varepsilon^{c(\rho-Vt)}$,

we readily find $M_1 = N_1 r = \rho\sin\theta\frac{dU}{d\rho}\left\{\sin\theta\frac{dV_{-n}}{dx} - \cos\theta\frac{dV_{-n}}{dr}\right\}.$

Putting
$$Q'_{n-1} = \frac{dQ_{n-1}}{d\mu},$$
and, omitting the constant term, we have
$$M_1 = \frac{dU}{d\rho}\frac{1}{\rho^n}\sin^2\theta Q'_{n-1}.$$
Hence we have, by replacing $n-1$ by n, and changing the sign of all the terms,
$$M_1 = -C_n\sin^2\theta\,Q'_n\,\varepsilon^{c(\rho-Vt)}.$$
From which we find, writing P for the radial component,
$$\Theta_1 = 0,$$
$$P_1 = 0,$$
$$N_1 = -\frac{C_n}{\rho}\sin\theta\,Q'_n\,\varepsilon^{c(\rho-Vt)},$$
$$\Theta_2 = \frac{c}{\rho(2n+1)}\{(n+1)C_{n-1} + nC_{n+1}\}\sin\theta\,Q'_n\,\varepsilon^{c(\rho-Vt)},$$
$$P_2 = -\frac{n(n+1)}{\rho^2}C_n\,Q_n\,\varepsilon^{c(\rho-Vt)},$$
$$N_2 = 0,$$
$$N_3 = -c^2 N_1.$$
We can also go backward and write
$$\Theta_0 = -\frac{\theta_2}{c^2},$$
$$P_0 = -\frac{P_2}{c^2},$$
$$M_{-1} = -\frac{M_1}{c^2}.$$

Hence, as I have remarked before, each of the vectors indicated by these components can be made equal to some one of the vectors used in the theory of light. Thus

ELECTRO-MAGNETIC THEORY.	ELASTIC SOLID THEORY.
Vector potential.	Linear displacement of particle.
Electric current or electro-motive force.	Rotation of particle.
Magnetic force, or induction, or magneto-motive force.*	Linear velocity of particle.
Rate of change of current, etc.	Angular velocity of particle, etc.
Rate of change of magnetic-force, etc.	

Should we add another term with $-i$ in the place of $+i$, we shall have the following real forms, since

$$\begin{aligned} C_n &= A_n - iB_n, \\ C'_n &= A_n + iB_n, \\ c &= a - ib, \\ c' &= a + ib. \end{aligned}$$

$$N_1 = -\frac{2\sin\theta\, Q'_n}{\rho}\,\varepsilon^{a(\rho-Vt)}\{A_n\cos b(\rho-Vt) - B_n\sin b(\rho-Vt)\},$$

$$\Theta_2 = \frac{2\sin\theta Q'_n}{\rho}\,\varepsilon^{a(\rho-Vt)}\left\{\left[A_{n-1} - \frac{nA_n}{\rho}\right]\cos b(\rho-Vt) - \left[B_{n-1} - \frac{nB_n}{\rho}\right]\sin b(\rho-Vt)\right\},$$

$$P_2 = -\frac{2n(n+1)}{\rho^2}\,Q_n\,\varepsilon^{a(\rho-Vt)}\{A_n\cos b(\rho-Vt) - B_n\sin b(\rho-Vt)\}.$$

And the other terms are readily obtained from these as follows:

$$\begin{aligned} N_{-1} = \frac{2\sin\theta Q'_n}{\rho(a-ib)^2}\,\varepsilon^{a(\rho-Vt)}\{&[(a^2-b^2)A_n + 2abB_n]\cos b(\rho-Vt) \\ &- [(a^2-b^2)B_n - 2abA_n]\sin b(\rho-Vt)\} \\ N_3 = \frac{2\sin\theta Q'_n(a-ib)^2}{\rho}\,\varepsilon^{a(\rho-Vt)}\{&[(a^2-b^2)A_n - 2abB_n]\cos b(\rho-Vt) \\ &- [(a^2-b^2)B_n + 2abA_n]\sin b(\rho-Vt)\}. \end{aligned}$$

As these quantities constantly return again to the same form with only a change in the constants, there will be only two cases, when the vector potential, or its analogy in the elastic solid theory, is made equal to the even or odd vectors. Let us take the first case, making the components of the vector potential equal to Θ_0 and P_0. The magnetic induction will then be N_1 and the components of the electric current

$$\frac{\theta_2}{4\pi\mu} \text{ and } \frac{P_2}{4\pi\mu}.$$

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* As Mr. Bosanquet has shown a disposition to claim this term, I may say that the idea expressed by it is of common occurrence in the works of Faraday, and I myself have used the term and given it mathematical expression in the American Journal of Mathematics. Mr. Bosanquet's article appeared in the Phil. Mag. for March, 1883.

Hence, in this case, the magnetic induction is in circles around the axis, and the electric currents in planes passing through the axis. At a great distance from the origin, the electric currents are on the sphere, but the normal component must still exist in order that the equation of continuity may be satisfied. The case of plane waves is the only one where the normal component entirely vanishes.

When we make the vector potential equal to N_1, the electric currents are in circles, and magnetic induction in planes passing through the axis.

Let us consider the first case. We shall find it much more convenient to continue with the imaginary form, remembering that a similar term with i changed in sign is always to be added except where it has already been done. We shall then have

Components of the Vector Potential

$$\Theta_0 = -\frac{\sin\theta\, Q'_n}{c(2n+1)\rho}\{(n+1)C_{n-1} + nC_{n+1}\}\varepsilon^{c(\rho-Vt)}.$$

$$P_0 = \frac{n(n+1)\,C_n}{c^2\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)}.$$

Magnetic Induction

$$N_1 = -\frac{C_n}{\rho}\sin\theta\, Q'_n\,\varepsilon^{c(\rho-Vt)}.$$

Components of the Electric Current

$$\frac{\theta_2}{4\pi\mu} = \frac{c\sin\theta\, Q'_n}{4\pi\mu(2n+1)\rho}\{(n+1)(C_{n-1}) + n\,C_{n+1}\}\varepsilon^{c(\rho-Vt)},$$

$$\frac{P_2}{4\pi\mu} = -\frac{n(n+1)\,C_n}{4\pi\mu\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)}.$$

Components of the Electric Displacement

$$-\frac{K}{4\pi}\frac{d\theta_0}{dt} = -\frac{VK\sin\theta\, Q'_n}{4\pi(2n+1)\rho}\{(n+1)\,C_{n-1} + n\,C_{n+1}\}\,\varepsilon^{c(\rho-Vt)},$$

$$-\frac{K}{4\pi}\frac{dP_0}{dt} = \frac{VKn(n+1)\,C_n}{4\pi c\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)},$$

where μ is the magnetic permeability and K the inductive capacity, and we have the well-known relation of Maxwell,

$$V^2 = \frac{1}{\mu K}.$$

Should a perfectly conducting sphere exist, the electric density on its surface will be

$$\sigma = -\frac{K}{4\pi\rho^2}\frac{d}{dt}\frac{dM_{-1}}{d\mu} = -\frac{K}{4\pi}\frac{dP_0}{dt},$$

$$\sigma = \frac{VKn(n+1)C_n}{4\pi c\rho^2} Q_n \varepsilon^{c(\rho-Vt)}.$$

This is the same as the radial component of the electric displacement at this point.

ARBITRARY DISTURBANCE.

In the equations of p. 368 make $a = 0$ and $n = 1$. We then have

$$C_1 = C_0\left\{1 - \frac{i}{b\rho}\right\},$$

$$C_2 = C_0\left\{1 - \frac{3i}{b\rho} - \frac{3}{b^2\rho^2}\right\}.$$

The magnetic induction is then

$$N'' = -C_1\frac{\sin\theta}{\rho}\varepsilon^{-ib(\rho-Vt)},$$

and the components of the electric displacement,

$$\Theta' = -\frac{VK\sin\theta}{12\pi\rho}(2C_0 + C_2)\varepsilon^{-ib(\rho-Vt)}$$

$$P' = +\frac{iVK\cos\theta}{2\pi b\rho^2}C_1\varepsilon^{-ib(\rho-Vt)}.$$

Let now a sphere of radius R be circumscribed about the origin and an arbitrary uniform displacement take place in the interior of this sphere of a value equal to

$$X'\varepsilon^{-ib(R-Vt)} = C_0\frac{iVK}{2\pi bR^2}\varepsilon^{-ib(R-Vt)}\left\{1 - \frac{i}{bR}\right\}.$$

Taking this value for the displacement inside the sphere and the previous values for the outside, the equation of continuity is satisfied for the electric displacement and for the magnetic induction. If the sphere is very small indeed, we have

$$X' = \frac{iVKC_0}{2\pi b^2R^3}.$$

Whence we have, on substituting the value of C_0 from this equation in the others and replacing $\frac{4}{3}\pi R^3$ by dv, the magnetic induction

$$N'' = -\frac{3b^2X'\sin\theta}{2VK\rho}\frac{C_1}{C_0}\varepsilon^{-ib(\rho-Vt)}dv,$$

and the electric displacement,

$$\Theta' = -\frac{2C_0 + C_2}{C_0}\,\frac{b^2X'}{8\pi\rho}\sin\theta\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$P' = \frac{C_1}{C_0}\,\frac{3ibX'}{4\pi\rho^2}\cos\theta\varepsilon^{-ib(\rho - Vt)}\,dv.$$

These equations give the complete solution of the problem of finding the disturbance at any point due to any arbitrary electrostatic disturbance throughout the medium. And it is to be noted that these equations are rigidly exact for all distances from the disturbance, and only have to be integrated to give the effect of any disturbance.

Had the disturbance been magnetic, we would have had for the magnetic induction,

$$\Theta'' = \frac{ib\sin\theta}{3\rho}\,(2C_0 + C_2)\,\varepsilon^{-ib(\rho - Vt)}$$

$$P'' = \frac{2\cos\theta}{\rho^2}\,C_1\varepsilon^{-ib(\rho - Vt)}.$$

And for the electric displacement,

$$N' = -\frac{iVKb\sin\theta}{4\pi\rho}\,C_1\varepsilon^{-ib(\rho - Vt)}.$$

Hence, taking a small sphere as before, the magnetic induction within it must be

$$X''\varepsilon^{-ib(R - Vt)} = \frac{2C_0}{R^2}\left(1 - \frac{i}{bR}\right)\varepsilon^{-ib(R - Vt)}$$

Whence, as before $\quad C_0 = -\frac{3b}{8\pi i}\,X''dv.$

in this case for the magnetic induction

$$\Theta'' = -\frac{X''ib^2\sin\theta}{8\pi\rho}\,\frac{2C_0 + C_2}{C_0}\,\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$P'' = \frac{3X''ib\cos\theta}{4\pi\rho^2}\,\frac{C_1}{C_0}\,\varepsilon^{-ib(\rho - Vt)}\,dv,$$

and for the electric displacement

$$N' = \frac{3b^2\,VKX''\sin\theta}{32\pi^2\rho}\,\frac{C_1}{C_0}\,\varepsilon^{-ib(\rho - Vt)}.$$

These equations give the complete solution of the disturbance throughout the medium due to an arbitrary magnetic disturbance at any point.

Although the disturbance is harmonic, yet we know by Fourier's theorem that any disturbance can be represented by a series of harmonic terms with the proper coefficients, and, indeed, we can replace the harmonic term by any function of $\rho - Vt$ or $\rho + Vt$. Should the disturbance not be parallel to the axis of X, we merely have to divide up the disturbance into its components and compute the effect of each and then add the components of the computed disturbance. In this way we readily find the effect of a general electric or magnetic disturbance.

Let the components of the arbitrary electric displacement be

$$\begin{array}{lcl} X'\varepsilon^{ibVt}, & & X''\varepsilon^{ibVt}, \\ Y'\varepsilon^{ibVt}, & \text{and of the magnetic induction} & Y''\varepsilon^{ibVt}, \\ Z'\varepsilon^{ibVt}, & & Z''\varepsilon^{ibVt}, \end{array}$$

where, in general, we must replace X', Y', Z', and X'', Y'', Z'' by a quantity of the complex form, and add another quantity with $-i$ in place of $+i$. Putting $D + iE$ for X', and adding the other term, we would have the real form

$$(D + iE)\varepsilon^{ibVt} + (D - iE)\varepsilon^{-ibVt} = 2\{D \cos bVt - E \sin bVt\},$$

which expresses the disturbance in any phase. But this is only necessary when we descend to actual calculation. The effect of this general disturbance is then found to be: The electric displacement

$$F' = \frac{b^2}{8\pi C_0\rho^3}\left\{X'(2C_0 + C_2)\rho^2 - 3S'C_2x - \frac{3KVC_1\rho}{4\pi}[Z''y - Y''z]\right\}\varepsilon^{-ib(\rho - Vt)}dv,$$

$$G' = \frac{b^2}{8\pi C_0\rho^3}\left\{Y'(2C_0 + C_2)\rho^2 - 3S'C_2y - \frac{3KVC_1\rho}{4\pi}[X''z - Z''x]\right\}\varepsilon^{-ib(\rho - Vt)}dv,$$

$$H' = \frac{b^2}{8\pi C_0\rho^3}\left\{Z'(2C_0 + C_2)\rho^2 - 3S'C_2z - \frac{3KVC_1\rho}{4\pi}[Y''x - X''y]\right\}\varepsilon^{-ib(\rho - Vt)}dv.$$

The magnetic induction,

$$F'' = \frac{b^2}{8\pi C_0\rho^3}\left\{X''(2C_0 + C_2)\rho^2 - 3S''C_2x + \frac{12C_1\pi\rho}{KV}[Z'y - Y'z]\right\}\varepsilon^{-ib(\rho - Vt)}dv;$$

$$G'' = \frac{b^2}{8\pi C_0\rho^3}\left\{Y''(2C_0 + C_2)\rho^2 - 3S''C_2y + \frac{12C_1\pi\rho}{KV}[X'z - Z'x]\right\}\varepsilon^{-ib(\rho - Vt)}dv,$$

$$H'' = \frac{b^2}{8\pi C_0\rho^3}\left\{Z''(2C_0 + C_2)\rho^2 - 3S''C_2z + \frac{12C_1\pi\rho}{KV}[Y'x - X'y]\right\}\varepsilon^{-ib(\rho - Vt)}dv,$$

where I have written

$$S' = X'x + Y'y + Z'z,$$
$$S'' = X''x + Y''y + Z''z.$$

For a general expression the term $\varepsilon^{-ib(\rho - Vt)}$ must be replaced by any function of $\rho - Vt$ or $\rho + Vt$ as before.

Before integration, one must of course substitute $(x - x')$, $(y - y')$ and $(z - z')$ for the x, y and z of the formula. It is evident that the components of the electric displacement can be replaced by the true convection displacement of electricity as carried along by the actual motion of the medium, and the disturbance due to a moving magnet can be calculated in a similar manner.

To obtain an idea of the relative magnitude of the quantities which enter into these expressions, I may remark that the value of b in wave lengths is $\frac{2\pi}{\lambda}$. Hence, for any ordinary calculations with respect to light, the distance need be only a few inches, or, indeed, one inch, to cause the values of C_1, C_2, etc. to become constant and equal to C_0. But if we are treating of longer waves we must retain such terms.

I have already given the proper expansion into series of the quantities here involved, but it will be better to put them in a reduced form,

$$c = -ib; \quad \rho = p\sqrt{1+2s}; \quad cps = q.$$

$$C_n(p) = C_0\left\{1 - \frac{n}{2}\,\frac{n+1}{cp} + \frac{n(n^2-1)}{2.4}\,\frac{n+2}{c^2p^2} - \frac{n(n^2-1)(n^2-2)}{2.4.6}\,\frac{n+3}{c^3p^3} + \text{etc.}\right\}$$

$$\frac{C_1(\rho)\,\varepsilon^{cp}}{\rho^2} = \frac{\varepsilon^{cp}}{p^2}\left\{C_1(p) + \frac{q}{1}\,C_2(p) + \frac{q^2}{1.2}\,C_3(p) + \text{etc.}\right\}$$

$$\frac{C_2(\rho)\,\varepsilon^{cp}}{\rho^3} = \frac{\varepsilon^{cp}}{p^3}\left\{C_2(p) + \frac{q}{1}\,C_3(p) + \frac{q^2}{1.2}\,C_4(p) + \text{etc.}\right\}$$

$$\frac{2C_0 + C_2(\rho)}{\rho}\,\varepsilon^{cp} = \frac{3\varepsilon^{cp}}{p}\left\{\frac{1}{3}\left(2C_0 + C_2(p)\right) + \text{etc.} \right.$$
$$\left. + \frac{q^n}{(2n+3)1.2.3\ldots n}\left[(2n+2)C_n(p) + C_{n+2}(p)\right] + \text{etc.}\right\}$$

Thus one has $\rho^2 = R^2 - 2(xx' + yy' + zz') + r^2$,

where $R^2 = x^2 + y^2 + z^2$ and $r^2 = x'^2 + y'^2 + z'^2$.

Hence one can write

$$p = R; \quad s = -\frac{xx' + yy' + zz'}{R^2} + \frac{r^2}{2R^2};$$

$$q = -c\,\frac{xx' + yy' + zz'}{R} + c\,\frac{r^2}{2R}.$$

When R is great and r is small, the last term will vanish and leave a very convenient form for many cases.

But it is not necessary to always perform the calculation at a finite distance, for we have the following theorem which I believe to be new.

Theorem. Supposing the sources of light to be continuous and knowing the light over a sphere at an infinite distance, it can be determined for all space by the following process:

Let F, G and H be the components of any of the vectors such as the electric displacement, etc., and suppose them known over the infinite sphere.

Express them in terms of series of surface harmonics, thus:

$$F = \frac{C_0 \varepsilon^{c(\rho - Vt)}}{\rho} \{ E_0 Y_0' + E_1 Y_1' + E_2 Y_2' + \text{ etc.} \}$$

$$G = \frac{C_0 \varepsilon^{c(\rho - Vt)}}{\rho} \{ E_0 Y_0'' + E_1 Y_1'' + E_2 Y_2'' + \text{ etc.} \}$$

$$H = \frac{C_0 \varepsilon^{c(\rho - Vt)}}{\rho} \{ E_0 Y_0''' + E_1 Y_1''' + E_2 Y_2''' + \text{ etc.} \}$$

At any other point of space the equations must satisfy the equations of light motion and also the equation of continuity. Referring to the equations of pp. 363 and 364, we see that if we multiply each term by the corresponding quantity C_n, the equations of light will be satisfied, and if the surface harmonics are of the form

$$Y_n' = \rho^n \left\{ y \frac{dV_{-(n+1)}}{dz} - z \frac{dV_{-(n+1)}}{dy} \right\},$$

$$Y_n'' = \rho^n \left\{ z \frac{dV_{-(n+1)}}{dx} - x \frac{dV_{-(n+1)}}{dz} \right\},$$

$$Y_n''' = \rho^n \left\{ x \frac{dV_{-(n+1)}}{dy} - y \frac{dV_{-(n+1)}}{dx} \right\};$$

the equation of continuity is also satisfied. Hence the components of the vector at any point of space are

$$F = \frac{\varepsilon^{c(\rho - Vt)}}{\rho} \{ C_0 E_0 Y_0' + C_1 E_1 Y_1' + C_2 E_2 Y_2' + \text{ etc.} \}$$

$$G = \frac{\varepsilon^{c(\rho - Vt)}}{\rho} \{ C_0 E_0 Y_0'' + C_1 E_1 Y_1'' + C_2 E_2 Y_2'' + \text{ etc.} \}$$

$$H = \frac{\varepsilon^{c(\rho - Vt)}}{\rho} \{ C_0 E_0 Y_0''' + C_1 E_1 Y_1''' + C_2 E_2 Y_2''' + \text{ etc.} \}$$

The best point for the origin will be somewhere near the center of gravity of the illuminated body.

That the light is perfectly determined in this way for all points outside a sphere around the origin which does not cut the illuminated body is evident from the fact that we might reverse the motions so as to make the light return along its previous path.

These expansions and this theorem entirely change the ideas of those who have only been in the habit of regarding light from the point of view of rays of light. For we here see that light coming from any source may be replaced by light from another source, so that the true source might be entirely invisible, and we "see" only the false source. This, however, could only happen for a point outside a sphere drawn around the false point and the real point, and we could always detect the deception by making a complete spherical journey around the real point and inside the false point.

As an illustration, let us compute the effect of a circular electric current which is caused to vibrate back and forth according to a simple harmonic function, so as to make the displacement $C_0\varepsilon^{-cVt}$. In this case we must make

$$0 = X' = X'' = Y'' = Z''; \quad Y'dv = -C_0 r \sin a da; \quad Z' = C_0 r \cos a da,$$

where r is the radius of the circle.

From the symmetry around the axis of x we have $F' = 0$, and can write for the component of the electric displacement around x at a distance $R \sin\theta$ from that axis,

$$N' = -\frac{rb^2\varepsilon^{-cVt}}{8\pi}\int_0^{2\pi}\left\{\frac{2C_0 + C_2}{\rho}\sin\alpha - \frac{3RrC_2\sin\theta}{\rho^3}\cos^2\alpha\right\}\varepsilon^{c\rho}\,d\alpha,$$

$$\rho^2 = R^2 - 2Rr\sin\theta\sin\alpha + r^2.$$

Integrating the first term by parts, taking $\sin a da$ for one part we have, since the first part disappears,

$$\int_0^{2\pi}\frac{2C_0 + C_2}{\rho}\varepsilon^{cs}\sin a da = \int_0^{2\pi}\frac{d}{da}\left(\frac{2C_0 + C_2}{\rho}\varepsilon^{cs}\right)\cos a da.$$

But
$$\frac{d}{da} = \frac{d\rho}{da}\frac{d}{d\rho} = -\frac{Rr\sin\theta\cos\alpha}{\rho}\frac{d}{d\rho}.$$

Whence
$$\frac{d}{da}\left(\frac{2C_0 + C_2}{\rho}\varepsilon^{c\rho}\right) = -Rr\sin\theta\cos\alpha\frac{3}{5}\frac{c\varepsilon^{c\rho}}{\rho^2}[4C_1 + C_3].$$

Whence we have

$$N' = \frac{3cRr^2b^2\sin\theta\varepsilon^{-cVt}}{8\pi}\int_0^{2\pi}\frac{C_1\varepsilon^{c\rho}}{\rho^2}\cos^2\alpha\,d\alpha.$$

In the expansion, put

$$p = R; \quad q = -cr\sin\theta\sin\alpha + c\frac{r^2}{2R}.$$

$$\frac{C_1\varepsilon^{c\rho}}{\rho_2} = \frac{\varepsilon^{cR}}{R^2}\left\{C_1(R) + \frac{q}{1}C_2(R) + \frac{q^2}{1.2}C_3(R) + \text{etc.}\right\}$$

Writing
$$g = \frac{cr^2}{2R} \text{ and } h = -cr\sin\theta,$$

we have
$$q = g + h\sin\alpha,$$

$$\int_0^{2\pi} q^n\cos^2\alpha\,d\alpha = 2\left\{\frac{1}{2}g^n + \frac{n(n-1)}{1.2}\frac{1}{2.4}g^{n-2}h^2 + \frac{n(n-1)(n-2)(n-4)}{1.2.3.4}\frac{1.3}{2.4.6}g^{n-4}h^4\right.$$
$$\left.+ \text{etc.} + \frac{n(n-1)\ldots(n-2m+1)}{12\ldots 2m}\frac{1.3\ldots(2m-1)}{2.4\ldots(2m+2)}g^{n-2m}h^{2m} + \text{etc.}\right\}$$

So that the problem is completely solved without any approximation and for all distances at which the series is convergent.

At a great distance the expression becomes very simple.

$$N' = \frac{3cr^2b^2C_0}{4\pi}\,\frac{\sin\theta}{R}\,\varepsilon^{c(R-Vt)}\left\{\frac{1}{2} + \frac{1}{2^2.4}\sigma^2r^2\sin^2\theta + \frac{1}{(2.4)^2 6}\,\sigma^4r^4\sin^4\theta \right.$$
$$\left. + \text{etc.} + \frac{1}{(2.4\ldots 2n)^2(2n+2)}\,c^{2n}r^{2n}\sin^{2n}\theta + \text{etc.}\right\}$$

We recognize this series as a Bessel's function divided by $br\sin\theta$, and so we can write

$$N' = -\frac{3ib^2rC_0}{4\pi R}\,\varepsilon^{c(R-Vt)}J_1(br\sin\theta).$$

This same series occurs in the expression for the light from a circular orifice, but I am not aware that writers on physical optics have recognized this connection with Bessel's functions. Prof. Stokes has given the value of the series for large values of $br\sin\theta$, and the same value is given by writers on Bessel's functions. We thus find, writing $v = br\sin\theta$

$$N' = -\frac{3ib^2rC_0}{4\pi R}\sqrt{\frac{2}{\pi v}}\,\varepsilon^{c(R-Vt)}\left[\begin{array}{l}\sin\left(v-\frac{\pi}{4}\right)\left\{1+\frac{3.5.1}{2.4}\frac{1}{(4v)^2}-\frac{3.5.7.9.1.3.5}{2.4.6.8}\frac{1}{(4v)^4}+\text{etc.}\right\}\\ +\cos\left(v-\frac{\pi}{4}\right)\left\{\frac{3}{2}\frac{1}{4v}-\frac{3.5.7.1.3}{2.4.6}\frac{1}{(4v)^3}+\text{etc.}\right\}\end{array}\right]$$

The energy given out per unit of time is, by the first formula,

$$\frac{3\,Vr^4b^6C_0^2}{2K}\left\{1-\frac{b^2r^2}{5}+\frac{b^4r^4}{35}-\frac{b^6r^7}{720}+\frac{b^8r^8}{31680}-\frac{b^9r^9}{1310400}+\text{etc.}\right\}$$

and by the second for very large values of br

$$\frac{Vrb^8C_0^2}{2\pi K}.$$

Let us now reduce the series to spherical harmonics.

Write

$$u = cr = -ibr = -\frac{2\pi ir}{\lambda},$$

$$P = u^2\left\{\frac{1}{2}+\frac{u^2}{2^2 4}\sin^2\theta+\frac{u^4}{(2.4)^2 6}\sin^4\theta+\text{etc.}\right\}$$

This satisfies the differential equation

$$\frac{d^2P}{du^2}-\frac{1}{u}\frac{dP}{du}-P\sin^2\theta = 0.$$

Write also $\mu = \cos\theta$ and $Q'_n = \frac{dQ_n}{d\mu}$. P can be developed in the following series as it only contains the even powers of μ;

$$P = A'_1 + A'_3Q'_3 + A'_4Q'_4 + \text{etc.}$$

But

$$Q'_n \sin^2\theta = -\frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}\{(2n-1)Q'_{n+2} - (4n+2)Q'_n + (2n+3)Q'_{n-2}\}.$$

Writing $\delta = \frac{d^2}{du^2} - \frac{1}{u}\frac{d}{du}$ we have

$$A'_3 = \frac{5.7}{3.4}\left\{\frac{4}{5}A'_1 - \delta A'_1\right\},$$

$$A'_5 = \frac{9.11}{5.6}\left\{2\frac{3.4}{5.9}A'_3 - \frac{2}{15}A'_1 - \delta A'_3\right\},$$

$$A'_7 = \text{etc.},$$

$$A'_n = \frac{(2n-1)(2n+1)}{n(n+1)}\left\{2\frac{(n-2)(n-1)}{(2n-5)(2n-1)}A'_{n-2} - \frac{(n-4)(n-3)}{(2n-5)(2n-7)}A'_{n-4} - \delta A'_{n-2}\right\}.$$

These all depend on A'_1, whose value can be found by developing $\sin^2\theta$, $\sin^4\theta$, etc., in harmonics. In the following values one must add the second term with $-i$ in place of $+i$ before substituting in the formula

$$A'_1 = \frac{3}{4}C_1(r)\,\varepsilon^{cr}$$

$$A'_3 = -\frac{21}{16}C_3(r)\,\varepsilon^{cr}$$

$$A'_5 = \frac{33}{10}\left\{\frac{1}{5}C_1(r) + \frac{5}{16}C_3(r)\right\}\varepsilon^{cr}$$

$$A'_7 = \text{etc.}$$

These reduce to Bessel's functions when we obtain the real form. Thus

$$A'_1 = \frac{3}{2}\left\{\frac{\sin br}{br} - \cos br\right\} = \frac{3}{2}\sqrt{\frac{\pi br}{2}}\,J_{\frac{3}{2}}(br).$$

Each term of the original series is now of the form $A'_n Q'_n \sin\theta$. But by the theorem of p. 372 we have only to multiply the terms of this by $C_n(R)$ and it gives the value for any finite distance outside a sphere around the origin which contains the circle. Hence

$$N' = \frac{3i\sin\theta}{4\pi R}\,\varepsilon^{c(R-Vt)}\{A'_1 C_0 + A'_3 Q'_3 C_3(R) + A'_5 Q'_5 C_5(R) + \text{etc.}\}.$$

DYNAMICAL THEORY OF DIFFRACTION.

When a ray of light strikes upon a screen in which there is an opening, a disturbance takes place in that opening, whose effect can be calculated by the preceding formulæ. Maxwell has shown, that in a plane wave, the energy is half magnetic and half electrostatic, and that the magnetic and electric displacements are at right angles to each other and in the same phase.

Stokes' solution is based upon the displacement and rate of displacement of his elastic medium. But in an elastic wave there is not only displacement but *rotation* also, and the components of this rotation must satisfy the equation of continuity. But when a wave is broken up at an orifice, the rotation is left discontinuous by Stokes' solution, and hence it cannot be exact. The equation of propagation of the rotation is the same as that of the displacement, and the two are at right angles to each other, and they are both equally important.

Hence, on the elastic solid theory as well as the electro-magnetic theory, the true solution of diffraction will depend upon the sums of two similar terms.

When we take account of both the terms, the fundamental properties by which Stokes attempted to obtain the direction of the motion of the particles vanishes, and that problem is impossible of solution by this means.

In the general equations of p. 371 let all the disturbances vanish except X' and Y'', so that the electric disturbance is in the direction of X and the magnetic in that of Y.

In order that the energy coming from the two disturbances may be equal we must have

$$\frac{V^2K}{8\pi}\left(\frac{3b^2X'}{2VK}\right)^2 = \frac{V^2K}{8\pi}\left(\frac{3b^2Y''}{8\pi}\right)^2,$$

or

$$Y'' = \frac{4\pi}{VK}X'$$

The electric displacement at any other point of space will then be

$$F' = \frac{b^2X'\nu}{8\pi C_0\rho^3}\{(2C_0 + C_2)\rho^2 - 3C_2x^2 + 3C_1\rho z\}\,\varepsilon^{-ib(\rho - Vt)}ds,$$

$$G' = \frac{b^2X'\nu}{8\pi C_0\rho^3}\{-3C_2xy\}\,\varepsilon^{-ib(\rho - Vt)}ds,$$

$$H' = \frac{b^2X'\nu}{8\pi C_0\rho^3}\{-3C_2xz - 3C_1x\rho\}\,\varepsilon^{-ib(\rho - Vt)}ds.$$

For showing the peculiarities of the case, polar coordinates are best. Let θ be the angle made by ρ with the axis of z and ϕ the angle around it from the plane XY. Let Θ', Θ'', Φ', Φ'', and P', P'' be the components of the electric displacement and magnetic induction to increase these angles and in the direction of ρ respectively. Then we have for the electric displacement

$$\Theta' = \frac{3b^2X'\nu}{8\pi\rho}\cos\phi\left\{[1 + \cos\theta]\left[1 - \frac{i}{b\rho}\right] - \frac{\cos\theta}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho - Vt)}ds,$$

$$\Phi' = -\frac{3b^2X'\nu}{8\pi\rho}\sin\phi\left\{[1 + \cos\theta]\left[1 - \frac{i}{b\rho}\right] - \frac{1}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho - Vt)}ds,$$

$$P' = \frac{3ibX'\nu}{4\pi\rho^2}\sin\theta\cos\phi\left\{1 - \frac{i}{b\rho}\right\}\varepsilon^{-ib(\rho - Vt)}ds.$$

And for the magnetic induction,

$$\Theta''=\frac{3b^2X'\nu}{2VK\rho}\sin\phi\left\{\left[1+\cos\theta\right]\left[1-\frac{i}{b\rho}\right]-\frac{\cos\theta}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi''=\frac{3b^2X'\nu}{2VK\rho}\cos\phi\left\{\left[1+\cos\theta\right]\left[1-\frac{i}{b\rho}\right]-\frac{1}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$P''=\frac{3ibX'\nu}{VK\rho}\sin\theta\sin\phi\left\{1-\frac{i}{b\rho}\right\}\varepsilon^{-ib(\rho-Vt)}ds.$$

In these, ν is the thickness of the disturbed stratum, so that $\nu ds = dv$. The electric displacement within the stratum is $X'\varepsilon^{ibVt}$.

When $b\rho$ is very large, the disturbance is in the spherical surface, and indeed it only requires a fraction of an inch from the origin to be able to omit the terms in $b\rho$, and also P' and P''. The equations then become very simple, as follows. The electric displacement,

$$\Theta'=\frac{3b^2X'\nu}{8\pi\rho}\cos\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi'=\frac{3b^2X'\nu}{8\pi\rho}\sin\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

and the magnetic induction,

$$\Theta''=\frac{3b^2X'\nu}{2VK\rho}\sin\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$P''=\frac{3b^2X'\nu}{2VK\rho}\cos\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds.$$

From either of these expressions or the more exact ones we see that the distribution of the magnetic induction is exactly the same, but turned around the axis of Z 90°, as the electric displacement. As this result would also apply to the elastic solid theory, we conclude that *diffraction gives no means of determining the relation between the displacement and plane of polarization.* Had we only taken the original disturbance as electrical alone, we would have arrived at Stokes' result.

Squaring the coefficients of the time function and adding, we find

$$\frac{9b^2X'^2\nu^2}{64\pi\rho^2}(1+\cos\theta)^2(ds)^2.$$

Hence the light is symmetrical around the axis of Z and varies from 4 in the positive direction to 1 in the plane XY and 0 in the negative direction.

It now remains to connect the arbitrary displacement $X'\nu$ with the intensity of the original wave.

Let the arbitrary displacement $X'\nu\varepsilon^{ibVt}$, with the corresponding magnetic quantity, exist throughout the plane XY. From considerations of symmetry, the electric displacement throughout space must be parallel to the axis X, and so we can write,

$$F' = \iint \frac{b^2 X'\nu}{8\pi C_0 \rho^3} \{(2C_0 + C_2)\rho^2 - 3C_2 x^2 + 3C_1 \rho z\} \varepsilon^{-ib(\rho - Vt)} ds,$$

$$z = \text{constant},$$

$$x = -r\cos\phi, \quad ds = rd\phi dr = \rho d\phi d\rho,$$

$$y = -r\sin\phi. \quad \rho^2 = z^2 + r^2.$$

The general and exact integral is

$$F' = \frac{3ibX'\nu}{8} \left\{ 1 + \frac{i + 2bz}{b\rho} + \frac{z^2}{\rho^2} - \frac{iz^2}{b\rho^3} \right\} \varepsilon^{-ib(\rho - Vt)}.$$

For positive values of z this gives, between the limits $\rho = \infty$ and $\rho = z$,

$$F = -\frac{3ibX'\nu}{2} \varepsilon^{-ib(z - Vt)}.$$

But for z negative it is zero. Hence such an arbitrary disturbance produces a wave in the positive direction, but none in the negative direction.

No approximation has been made in obtaining this quantity, and it evidently applies to a plane of any size, even infinitessimal, provided the point under consideration is infinitely near to it. The displacement near the surface therefore differs in phase $\frac{1}{2}\pi$ from the arbitrary disturbance, but is dependent upon its value at that particular point.

Hence we can replace any particular wave surface by a surface of arbitrary disturbance whose phase differs $\frac{1}{2}\pi$ from that of the wave. Such a surface of arbitrary disturbance then produces the same effect at all points of space as the original wave. Should the wave be spherical and of short radius, the normal component will enter and complicate the result, though the solution can be obtained but would evidently be complicated.

But if the wave have a radius of an inch or even less, the displacement is practically perpendicular to the radius, and the solution here given will apply.

Let l, m, n be the direction cosines of the normal to such a wave, and l', m' and n' the direction cosines of the electric displacement which is represented by

$$I\varepsilon^{-ib(R - Vt)}.$$

Then the values of the arbitrary displacement and magnetic induction to substitute in the general equations of p. 371, to produce such a wave will evidently be

$$X'\nu = \frac{2i}{3b} Il', \quad X''\nu = \frac{8\pi i}{3\sqrt{Kb}} Il'',$$

$$Y'\nu = \frac{2i}{3b} Im', \quad Y''\nu = \frac{8\pi i}{3\sqrt{Kb}} Im'',$$

$$Z'\nu = \frac{2i}{4b} In', \quad Z''\nu = \frac{8\pi i}{3\sqrt{Kb}} In'',$$

Where l'', m'', n'' are the direction cosines of the magnetic induction. We also have

$$ll' + mm' + nn' = 0,$$
$$ll'' + mm'' + nn'' = 0,$$
$$l'l'' + m'm'' + n'n'' = 0.$$

Returning again to the equations of p. 377, we see that the polarized light is diffracted equally in all directions from a very small orifice and independent of its plane of polarization. Furthermore the plane of polarization at any point is found by drawing a sphere through that point with its center at the orifice, and then drawing a plane through the given point and the point where the incident light first cuts the sphere, and cutting the plane of polarization of the incident light in a line perpendicular to the incident ray. The intersection of the plane and sphere then give the direction of the polarization.

It is seen that in both these particulars my solution differs from that of Prof. Stokes, and the construction is the same whether one takes the electric or magnetic quantities as the direction of polarization.

The system of planes for the electric and magnetic quantities form a system of orthogonal circles on the sphere.

The following constructions can also be used for obtaining the direction of the electric and magnetic quantities. Draw a sphere around the orifice and draw an axis through the sphere in the direction of the incident light. Then rule a sheet of paper with a series of lines at right angles to each other. Cut a star shaped piece out of the paper with its diameter equal to the circumference of the sphere, and having a very large number of points. Place the center of the star on the sphere at the end of the axis where the light leaves the sphere, and wrap the points around the sphere, the points meeting around the incident ray. The marks on the paper then give the required directions.

We can also construct the curves of polarization by noting that the stereographic projection of the lines on a plane is merely a series of straight lines.

The equations become very simple at many wave lengths distance from the orifice, especially when the orifice is small. If the radius of the original wave is large, it is usually sufficient to consider the periodic factor as the only variable. In this case we can write

$$\Theta' = \frac{ibI}{4\pi\rho}\cos\phi\,[1+\cos\theta]\iint \varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi' = -\frac{ibI}{4\pi\rho}\sin\phi\,[1+\cos\theta]\iint \varepsilon^{-ib(\rho-Vt)}ds,$$

$$P' = 0,$$

where I is the coefficient of the original vibration, and therefore its square is the intensity of the original light. The intensity of the diffracted light is simply proportional to the sum of the squares of Θ' and Φ'. This is the expression ordinarily used except the term in θ. Thus for a circular orifice we have the vibration expressed in Bessel's functions,

$$\Theta' = \frac{iIr}{2R}\cos\phi\,(1+\cos\theta)\,\frac{J_1(br\sin\theta)}{\sin\theta}\,\varepsilon^{-ib(R-Vt)},$$

$$\Phi' = -\frac{iIr}{2R}\sin\phi\,(1+\cos\theta)\,\frac{J_1(br\sin\theta)}{\sin\theta}\,\varepsilon^{-ib(R-Vt)}.$$

It is impossible to pass from these expressions to the case of a plane wave since they are only for the case of a great distance from a small orifice.